

HORIZON REGULARITY AND DILATON COUPLING QUANTIZATION IN EMD DYONS

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Four-dimensional Einstein-Maxwell-Dilaton theory admits asymptotically flat extremal dyonic solutions for an infinite discrete sequence of the coupling constant values. The quantization condition arises as consequence of regularity of the dilaton function at the event horizon. These dyons satisfy the no-force condition and have flat reduced three spaces like true BPS configurations, but no supersymmetric embeddings are known except for some cases of lower values of the coupling sequence.

Keywords: Dilaton gravity, black holes, BPS bound.

1. Introduction

Charged black holes in four-dimensional Einstein-Maxwell-dilaton (EMD) gravity exhibit different features depending on the value of the dilaton coupling constant a entering the Maxwell term $e^{-2a\phi}F^2$ in the Lagrangian. Several particular values of a have higher-dimensional supergravity origin. For $a = 0$ the dilaton decouples and the theory reduces to the Einstein-Maxwell (EM) system, which is the bosonic part of $N = 2$, $D = 4$ supergravity. In this case the extremal dyons, defined geometrically as black holes with the degenerate event horizon, saturate the supergravity Bogomol'nyi-Prasad-Sommerfield (BPS) bound, and have $AdS_2 \times S^2$ horizon with zero Hawking temperature. For $a \neq 0$ only the static purely electric or magnetic black are known analytically¹⁻⁴. These solutions generically have two horizons, the inner one being singular, so in the extremality limit the event horizon becomes a null singularity with vanishing Beckenstein-Hawking entropy and finite temperature (small black holes). For $a = (p/(p+2))^{1/2}$ small black holes may be interpreted as compactified regular non-dilatonic p -branes in $(4+p)$ -dimensional EM theory⁵. The value $a = 1$ corresponds to $N = 4$, $D = 4$ supergravity or dimensionally reduced heterotic string effective action, in this case the dyonic solutions are also known^{1,2} which are non-singular in the extremal limit and possess $AdS_2 \times S^2$ horizons. The last particular case, $a = \sqrt{3}$, corresponds to dimensionally reduced $N = 2$, $D = 5$ supergravity; in this case the static dyon solutions also have the $AdS_2 \times S^2$ horizon structure in the extremality limit.

The rotating dyonic solutions are known analytically only for $a = 0$ and $a = \sqrt{3}$ ^{6,7}. In the first case it is the Kerr-Newman solution of the EM theory, while in the second these were derived using the three-dimensional sigma-model on the symmetric space $SL(3, R)/SO(2, 1)$, corresponding to vacuum five-dimensional gravity. EMD theories with these two values of the dilaton coupling exhaust the set of models reducing to three-dimensional sigma-models on coset spaces⁸, so from this reasoning there are no indications on any particular status of EMD theories with

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other a . Meanwhile, as was shown numerically by Poletti, Twamley and Wiltshire⁹, the values $a = 0, \sqrt{3}$ are just the two lowest members $n = 1, 2$ of the “triangular” sequence of dilaton couplings

$$a_n = \sqrt{n(n+1)}/2, \quad (1)$$

selecting EMD theories in which numerical non-extremal dyonic solutions exist with two horizons and admit the extremal limit.

Some attempts are known to explore possibility of supersymmetric embedding of the EMD theory with arbitrary a . Using Witten-Nester construction, Gibbons et al.¹¹, were able to derive the BPS-like inequality for arbitrary a . Meanwhile, as was later shown by Nozawa and Shiromizu¹², the corresponding Killing spinor equations do not imply the initial bosonic equations as the integrability condition, unless $a = 0, \sqrt{3}$. So the question whether there is some supersymmetry underlying the rule (1) still remains open.

2. The setup

We choose the Einstein-Maxwell-dilaton lagrangian in the form

$$L = R - 2(\partial\phi)^2 - e^{-2a\phi}F^2, \quad (2)$$

and assume the static ansatz for the metric and the Maxwell one-form :

$$ds^2 = -e^{2\delta} N dt^2 + \frac{dr^2}{N} + R^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (3)$$

$$A = -f dt - P \cos\theta d\phi, \quad (4)$$

where P is the magnetic charge. The functions f, ϕ, N, R and δ depend on the radial variable r only. The equations of motions can be readily found from the reduced one-dimensional lagrangian

$$\mathcal{L} = 2ke^\delta + 2e^{-\delta} (e^{2\delta}NR)'R' - 2\phi'^2 e^\delta NR^2 + 2e^{-2a\phi} \left(f'^2 R^2 e^{-\delta} - \frac{P^2 e^\delta}{R^2} \right) \quad (5)$$

Two particular gauges are relevant: one is $\delta = 0$ in which some exact solutions are known, another is $R = r$, which is suitable for analytic derivation of the quantization rules and for numerical calculations. Imposing the coordinate condition and solving the Maxwell equations in the gauge $R = r$, we find:

$$f' = \frac{Q e^{\delta+2a\phi}}{r^2},$$

where Q is the integration constant (the electric charge). The remaining dilaton and Einstein's equations then read:

$$(N\phi' e^\delta r^2)' = \frac{2ae^\delta |PQ|}{r^2} \sinh(2a\phi), \quad (6)$$

$$\delta' = \phi'^2 r, \quad (7)$$

$$e^{-\delta} (e^\delta N r)' = 1 - \frac{2|PQ| \cosh(2a\phi)}{r^2}, \quad (8)$$

where the shifted dilaton function is introduced

$$\varphi = \phi - (\ln z)/2a, \quad z = \left| \frac{P}{Q} \right|.$$

The third Einstein equation is related to (7-8) via the Bianchi identity. This system of equations possess a discrete electric-magnetic duality

$$P \leftrightarrow Q, \quad \varphi \leftrightarrow -\varphi. \quad (9)$$

Asymptotic flatness (AF) implies $N(\infty) = 1$, $\delta(\infty) = 0$ with the next to leading terms

$$N \sim 1 - \frac{2M}{r} + (2|QP| \cosh(2a\varphi) + \Sigma^2) \frac{1}{r^2}, \quad (10)$$

$$e^\delta \sim 1 - \frac{\Sigma^2}{2r^2}, \quad (11)$$

$$\phi \sim \phi_\infty + \frac{\Sigma}{r}, \quad (12)$$

where M , Σ and ϕ_∞ are free parameters of the local series solution. As expected, for global solutions the dilaton charge Σ is not an independent quantity: integrating the Eq. (6) from r_h to ∞ , one obtains the sum rule:

$$\Sigma = 2a|QP| \int_{r_h}^{\infty} \frac{e^\delta \sinh 2a\varphi}{r^2} dr. \quad (13)$$

It can be shown that for the AF solutions with the degenerate horizon there is a second constraint on the charges, namely the no-force condition. First, from the equations of motion one can deduce that the quadratic form

$$I = \left(\frac{1}{2} N^2 e^{2\delta} \varphi'^2 + \frac{1}{8} e^{-2\delta} (N e^{2\delta})'^2 \right) r^4 - |QP| N e^{2\delta} \cosh(2a\varphi) \quad (14)$$

is conserved on shell: $\frac{dI}{dr} = 0$, (similar expression in the gauge $\delta = 0$ was given in⁹). Substituting here $r = r_h$, which solves two equations $N = 0 = N'$ in the case of the degenerate horizon, one finds that this integral has zero value $I = 0$. Then substituting the asymptotic expansions (10-12) we obtain from $I = 0$ the no-force condition:

$$M^2 + \Sigma = Q_\infty^2 + P_\infty^2, \quad (15)$$

where $Q_\infty = Q e^{2a\phi_\infty}$, $P_\infty = P e^{-2a\phi_\infty}$.

3. Exact solutions

An exact extremal dyon solution with $a = 1$ in this gauge reads:

$$\begin{aligned} e^{-2\delta} &= 1 + \frac{\Sigma^2}{r^2}, \quad e^{2\varphi} = \left| \frac{Q}{P} \right| \cdot \frac{\sqrt{r^2 + \Sigma^2} + \Sigma}{\sqrt{r^2 + \Sigma^2} - \Sigma}, \\ N &= \left(1 - \frac{2M}{r^2} \sqrt{r^2 + \Sigma^2} + \frac{Q^2 + P^2 + \Sigma^2}{r^2} \right), \end{aligned} \quad (16)$$

It has an event horizon at $r_h = \sqrt{M^2 - \Sigma^2}$ and the dilaton charge satisfying $2M\Sigma = P^2 - Q^2$. This solution saturates the fake BPS bound¹¹:

$$M \geq \frac{1}{\sqrt{1+a^2}} \sqrt{Q^2 + P^2}. \quad (17)$$

Another exact dyon solution is known^{1,2} for $n = 2$, ($a = \sqrt{3}$). The corresponding dilaton charge satisfies an interpolating formula

$$\frac{Q^2}{\Sigma - aM} + \frac{P^2}{\Sigma + aM} = \frac{1+a^2}{2a^2} \Sigma, \quad (18)$$

together with (3) (this is valid only for $a = a_n, n = 1, 2$).

Further information about higher n dyons can be extracted from known exact solutions for singly charged black holes. This provides us with some knowledge about the limiting point $z = 0$, an opposite limit $z = \infty$ can be explored via the discrete electric-magnetic duality (9)). These solutions are simpler in the gauge $\delta = 0$, the corresponding equations of motion being

$$(N\varphi'R^2)' = \frac{2a|PQ|}{R^2} \sinh(2a\varphi), \quad (19)$$

$$R'' + \varphi'^2 R = 0, \quad (20)$$

$$(NR)' = k - \Lambda R^2 - \frac{2|PQ| \cosh(2a\varphi)}{R^2}. \quad (21)$$

The electrically charged solution valid for all a in the general non-extremal case reads:

$$R^2 = \rho^2 f_-^{1-\gamma}, \quad N = f_+ f_-^\gamma, \quad e^{2a\varphi} = f_-^{\frac{2a}{1+a^2}}, \quad \gamma = \frac{1-a^2}{1+a^2}, \quad (22)$$

with $f_\pm = 1 - r_\pm/\rho$, where we denoted the radial coordinate as ρ to distinguish it from r in the gauge used in the Eqs. (6-8). The mass, the electric charge and the dilaton charge are related to r_\pm via

$$r_+ r_- = \frac{2Q^2}{1+\gamma}, \quad 2M = r_+ + \gamma r_-, \quad \Sigma = -\frac{a}{\Delta} r_-. \quad (23)$$

Note that our system of equations has special points of two kinds: the zeroes of the function $R(\rho)$, which correspond to the curvature singularity, and the zeroes of $N(\rho)$, for which $R \neq 0$; these are regular points. Generic behavior of the metric functions and the dilaton near the curvature singularity is non-analytic, while in the second case it is analytic. This is clearly seen in the solution (22) which has two zeroes of $N(\rho)$ at $\rho = r_\pm$, with r_- being a zero of $R(\rho)$ too, i.e., singular. The dilaton is analytic at the regular horizon $\rho = r_+$, but non-analytic in the singularity $\rho = r_-$. This can be expected for more general dyon solutions as well.

In the extremal limit $r_+ = r_-$ the dilaton is therefore singular at the horizon, but this does not influence its asymptotic behavior. So the dilaton charge is still finite and given by (23). In the extremal limit it reads:

$$\Sigma = -\frac{aQ}{\sqrt{\Delta}}. \quad (24)$$

This expression is expected to match the corresponding limit of dyonic solutions.

4. Coupling quantization

Now let us look for extremal dyon solutions with the regular horizon for general dilaton coupling constant. It turns out that the *local* power series solution valid in the vicinity of the degenerate horizon implies a constraint on the dilaton coupling constant. Coming back to the gauge $R = r$, in which the curvature singularity is at $r = 0$, and assuming that the horizon radius r_h satisfying $N(r_h) = N'(r_h) = 0$ is finite, we will have in the leading order:

$$N = \nu x^2 + O(x^3), \quad x = (r - r_h)/r_h, \quad (25)$$

$$\varphi = \varphi_h + \mu x^n + O(x^{n+1}), \quad (26)$$

where μ and ν are dimensionless parameters, and n is an integer. Substituting this into the Eq. (7) we find

$$\delta = \delta_h + \frac{\mu^2 n^2}{(2n-1)} x^{2n-1} + O(x^{2n}), \quad (27)$$

and therefore e^δ is finite and continuous at the horizon. So the leading term of the l.h.s. of the Eq. (6) is

$$(N\varphi' e^\delta r^2)' = \nu \mu n(n+1) e^{\delta_h} r_h^2 x^n + O(x^{n+1}). \quad (28)$$

This is zero at $x = 0$ for any n , so looking at the r. h. s. of Eq. (6) we immediately find that $\varphi_h = 0$. Then the linear in x term at the r. h. s. of (6) agrees with that at the l.h.s. and we find:

$$\nu n(n+1) = 4a^2 |QP|/r_h^2. \quad (29)$$

Now consider the equation (8) in the vicinity of $r \sim r_h$. One sees that the l.h.s. is linear in x , so the r.h.s. has to be linear in x too, leading to an equation for r_h :

$$r_h^2 = 2|QP|, \quad (30)$$

while expanding $r^{-2} = r_h^{-2}(1-2x) + \dots$ and equating the linear in x terms we obtain $\nu = 1$. Substituting this and (30) into (29), we arrive at

$$a^2 = \frac{n(n+1)}{2}. \quad (31)$$

This is the necessary condition for existence of the AF regular extremal dilatonic dyons which coincides with (1). It shows, in particular, that such solutions do not exist for $a < 1$. Numerical integration show that dyons with higher n do exist as global solutions indeed¹⁶. This selects the class of EMD theories with discrete dilaton couplings. Their supersymmetric origin (if any) is still enigmatic.

The set of EMD theories admitting extremal dyons exist in any dimensions and admit generalizations with negative cosmological constant Λ . In this case the condition on the dilaton coupling contains one integer, one continuous parameter

(Λ), and the topological parameter $k = 0, \pm 1$ standing for plane, spherical and hyperbolic topologies.

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